J. Fluid Mech. (1997), *vol.* 347, *pp.* 119–139. Printed in the United Kingdom

Nonlinear diffusive surface waves in porous media

By PHILIP L.-F. LIU AND JIANGANG WEN†

School of Civil and Environmental Engineering, Cornell University, Ithaca, NY 14853, USA

(Received 22 October 1996 and in revised form 20 April 1997)

A fully nonlinear, diffusive, and weakly dispersive wave equation is derived for describing gravity surface wave propagation in a shallow porous medium. Darcy's flow is assumed in a homogeneous and isotropic porous medium. In deriving the general equation, the depth of the porous medium is assumed to be small in comparison with the horizontal length scale, i.e. $O(\mu^2) = O(h_0/L)^2 \ll 1$. The order of magnitude of accuracy of the general equation is $O(\mu^4)$. Simplified governing equations are also obtained for the situation where the magnitude of the free-surface fluctuations is also small, $O(\epsilon) = O(a/h_0) \ll 1$, and is of the same order of magnitude as $O(\mu^2)$. The resulting equation is of $O(\mu^4, \epsilon^2)$ and is equivalent to the Boussinesq equations for water waves. Because of the dissipative nature of the porous medium flow, the damping rate of the surface wave is of the same order magnitude as the wavenumber. The tide-induced groundwater fluctuations are investigated by using the newly derived equation. Perturbation solutions as well as numerical solutions are obtained. These solutions compare very well with experimental data. The interactions between a solitary wave and a rectangular porous breakwater are then examined by solving the Boussinesq equations and the newly derived equations together. Numerical solutions for transmitted waves for different porous breakwaters are obtained and compared with experimental data. Excellent agreement is observed.

1. Introduction

The Dupuit approximation, assuming that the flow is essentially horizontal and the pressure field hydrostatic, is a good approximation in many applications for groundwater flows in an unconfined aquifer (e.g. Bear 1972). The Dupuit approximation is the same as the shallow-water approximation commonly used in studying surface water waves or open channel flows. These approximations are valid as long as the horizontal scale of flow is much larger than the vertical scale. Combination of the Dupuit approximation and Darcy's law for a saturated flow in porous media results in the well-known 'Boussinesq equation' for unsteady flow in a phreatic aquifer (Bear 1972)

$$\nabla \cdot (KH\nabla H) = n_e \frac{\partial H}{\partial t}, \tag{1.1}$$

in which H is the thickness of the aquifer, K the depth-averaged permeability, n_e the effective porosity and ∇ the horizontal gradient. This nonlinear diffusion equation

† Present address: Ocean Engineering Program, Department of Civil Engineering, Texas A & M University, College Station, TX 77843-3136, USA.

has been used as the mathematical model for describing many groundwater flow phenomena, including the tide-induced groundwater table fluctuations (e.g. Philip 1973; Knight 1982; Nielsen 1990; Nielsen *et al.* 1996).

In an attempt to include the effects of vertical acceleration, Dagan (1967) and Parlange *et al.* (1984) derived a set of perturbation equations with the small parameter characterizing the shallowness of the aquifer. While their leading-order equations remain the same as (1.1), Dagan's second-order equations, forced by the leading-order solutions, contained some typographical errors. However, those typographical errors did not affect the remainder of Dagan's paper. In both studies the bottom of the aquifer was assumed to be flat. Parlange *et al.* limited their analysis to one-dimensional periodic flows.

In this paper we rederive the flow equations in a shallow phreatic porous medium by using two parameters

$$\epsilon = \frac{a}{h_0}, \qquad \mu^2 = \left(\frac{h_0}{L}\right)^2, \tag{1.2}$$

where a represents the phreatic surface displacement measured from the still water level, h_0 the characteristic aquifer thickness and L the characteristic horizontal scale of the aquifer. In the general derivation, the parameter representing the shallowness of the aquifer, μ^2, is assumed to be small, while the parameter for nonlinearity, ϵ, remains $O(1)$. The resulting equations are truncated and are accurate up to $O(\mu^4)$. The general equations are two-dimensional and the porous medium thickness can vary in the horizontal dimensions.

A set of simplified equations is derived based on the 'Boussinesq approximation', which assumes that the nonlinearity parameter, ϵ, is of the same order of magnitude as μ^2. The characteristics of these equations are studied extensively for the constant-depth case. Basically solutions to the governing equations represent nonlinear diffusive waves for which the spatial damping rate is of the same order magnitude as the wavenumber. The inclusion of the $O(\mu^2)$ term modifies not only the phase (wavenumber), but also the damping rate.

One-dimensional equations are then used to study two practical problems: (i) the tide-induced free-surface fluctuations and (ii) the transmission and reflection of solitary waves by a porous breakwater. For the first problem, analytical perturbation solutions as well as numerical solutions are obtained with tidal effects being introduced through the boundary condition. Analytical and numerical results are compared with experimental data. Good agreement is observed. Inclusion of the higher-order term, $O(\mu^2)$, improves the agreement significantly. In the second problem numerical solutions are obtained for the interactions between an incident solitary wave and a rectangular porous breakwater. The conventional Boussinesq equations are used to calculate the incident, reflected and transmitted solitary waves. Inside the porous breakwater a linearization process is used to convert the nonlinear resistance formula to a Darcy-type resistance (e.g. Madsen 1974). Numerical solutions are compared with experimental data (Vidal *et al.* 1988). Excellent agreement is observed.

This paper is organized in the following manner. In § 2 the formulation of a three-dimensional Darcy flow with a free surface is reviewed first. The general derivation of the governing equations for fully nonlinear long waves is given in § 3. We show that Parlange *et al.*'s (1984) perturbation equations are special cases of our general equations. The Boussinesq approximation is introduced into the general governing equations in § 4. The basic characteristics of the simplified equation are investigated. Analytical and numerical solutions are presented in § 5 and § 6, respectively, for both tide-induced groundwater fluctuations and the interactions between solitary waves and a porous breakwater.

2. Darcy's flows and boundary conditions

The free-surface flow of an incompressible fluid in a rigid, homogeneous and isotropic porous medium is considered here. The flow obeys Darcy's law so that the seepage velocity is proportional to the gradient of the piezometric head, i.e.

$$\boldsymbol{u} = -K\nabla_3 \Phi, \tag{2.1}$$

where

$$\Phi = \frac{P}{\gamma} + z, \tag{2.2}$$

P is the pressure, $\gamma = \rho g$ the specific weight, K the permeability, z the vertical axis, and ∇_3 the three-dimensional gradient vector. In the vertical direction the flow domain is bounded by an impermeable boundary, $z = -h(x, y)$, and a free surface, $z = \zeta(x, y, t)$. The continuity requires that $\nabla_3 \cdot \boldsymbol{u} = 0$. From (2.1) we obtain

$$\nabla_3^2 \Phi = 0, \qquad -h < z < \zeta, \tag{2.3}$$

for a homogeneous medium, K = constant. On the free surface, the pressure is a constant which can be assigned as zero. Thus, from (2.2)

$$\Phi = \zeta \qquad \text{on } z = \zeta. \tag{2.4}$$

The kinematic free-surface boundary condition requires

$$\frac{\partial \zeta}{\partial t} - \frac{K}{n_e}(\nabla \Phi \cdot \nabla \zeta) + \frac{K}{n_e}\frac{\partial \Phi}{\partial z} = 0 \quad \text{on } z = \zeta, \tag{2.5}$$

where n_e is the effective porosity and $\nabla = \left(\partial/\partial x, \partial/\partial y\right)$ the horizontal gradient. Along the impervious bottom, $z = -h$, the normal flux vanishes:

$$\frac{\partial \Phi}{\partial z} = -\nabla \Phi \cdot \nabla h \qquad \text{on } z = -h. \tag{2.6}$$

Introducing L as the horizontal length scale of the problem and h_0, the typical depth, as the vertical scale and a as the scale of the free-surface displacement, we can normalize the variables as follows:

$$\zeta \to a\zeta, \quad z \to h_0 z, \quad (x, y) \to L(x, y), \tag{2.7a}$$

$$t \to \frac{L^2 n_e}{K h_0} t, \qquad \Phi \to a\Phi. \tag{2.7b}$$

The resulting dimensionless governing equation and boundary conditions can be written as

$$\mu^2 \nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0, \qquad -h < z < \epsilon\zeta, \tag{2.8a}$$

$$\Phi = \zeta \qquad \text{on } z = \epsilon\zeta, \tag{2.8b}$$

$$\mu^2 \left[\frac{\partial \zeta}{\partial t} - \epsilon \nabla \Phi \cdot \nabla \zeta\right] + \frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } z = \epsilon\zeta, \tag{2.8c}$$

$$\frac{\partial \Phi}{\partial z} + \mu^2 \nabla \Phi \cdot \nabla h = 0 \qquad \text{on } z = -h, \tag{2.8d}$$

where two parameters, μ^2 and ϵ, have been defined in (1.2).

An alternative way of scaling the problem is to specify the time scale as T. The length scale L can be found from (2.7) as

$$L = \left(\frac{K h_0 T}{n_e}\right)^{1/2}. \tag{2.9}$$

If one is interested in examining the response of a coastal phreatic aquifer to tidal fluctuations, the following parameters can be used: $K = 10^{-2}$–10^{-4} m s^{-1}, $n_e = O(10^{-1})$, $h_0 = O(10\text{ m})$, $T = 12$ hr. From (2.9) the length scale is about 20–200 m. Therefore, the parameter μ^2 is of order of magnitude $O(10^{-1}$–$10^{-2})$. In this paper, we consider a shallow aquifer such that $O(\mu^2) \ll 1$. If the tidal range is of order of magnitude 1 m, i.e. $a = O(1\text{ m})$, the parameter ϵ is also of order of magnitude $O(10^{-1})$. However, to find general equations, the parameter ϵ is assumed to be an order one quantity.

We remark here that the boundary-value-problem given in (2.8) is the same as the one derived for water waves with the exception of the dynamic free-surface boundary condition (2.8*b*). The depth-integrated continuity equation can be derived by integrating (2.8*a*) from $-h$ to $\epsilon\zeta$ and using the boundary conditions. Thus

$$-\nabla \cdot \int_{-h}^{\epsilon\zeta} \nabla\Phi \, \mathrm{d}z + \frac{\partial \zeta}{\partial t} = 0, \tag{2.10}$$

which is an exact equation.

3. Perturbation solutions for shallow water

For small μ^2 we seek for the following perturbation solutions:

$$\Phi(x, y, z, t) = \sum_{n=0}^{\infty} \mu^{2n} \Phi_n(x, y, z, t). \tag{3.1}$$

Using the Laplace equation (2.8*a*) and the bottom boundary condition (2.8*d*), we can express the preceding equation up to $O(\mu^2)$ in terms of $\Phi_\alpha(x, y, t) = \Phi(x, y, z_\alpha(x, y), t)$, in which $z_\alpha(x, y)$ is a prescribed surface. Thus

$$\Phi(x, y, z, t) = \Phi_\alpha(x, y, t) + \mu^2 \left[(z_\alpha - z)\nabla \cdot (h\nabla\Phi_\alpha) + \tfrac{1}{2}\left(z_\alpha^2 - z^2\right)\nabla^2\Phi_\alpha\right] + O(\mu^4). \tag{3.2}$$

The procedures for obtaining the equation above have been given in Chen & Liu (1995) for the water wave problem and will not be repeated here. The corresponding horizontal and vertical velocity can be expressed as

$$-\nabla\Phi = -\nabla\Phi_\alpha - \mu^2\nabla\left[(z_\alpha - z)\nabla \cdot (h\nabla\Phi_\alpha) + \tfrac{1}{2}\left(z_\alpha^2 - z^2\right)\nabla^2\Phi_\alpha\right] + O(\mu^4), \tag{3.3a}$$

$$-\frac{\partial\Phi}{\partial z} = \mu^2\left\{\nabla \cdot (h\nabla\Phi_\alpha) + z\nabla^2\Phi_\alpha\right\} + O(\mu^4). \tag{3.3b}$$

Similarly to the long water wave problem, the leading-order vertical velocity is of $O(\mu^2)$ and is linear in z, while the leading-order horizontal velocity is uniform in the z-direction.

Substituting (3.2) into the continuity equation (2.10), we obtain

$$\frac{\partial\zeta}{\partial t} - \nabla \cdot [(\epsilon\zeta + h)\nabla\Phi_\alpha] - \mu^2\nabla \cdot \Big\{(\epsilon\zeta + h)\nabla\left[z_\alpha\nabla \cdot (h\nabla\Phi_\alpha) + \tfrac{1}{2}z_\alpha^2\nabla^2\Phi_\alpha\right] - \tfrac{1}{2}\left(\epsilon^2\zeta^2 - h^2\right)\nabla\left[\nabla \cdot (h\nabla\Phi_\alpha)\right] - \tfrac{1}{6}\left(\epsilon^3\zeta^3 + h^3\right)\nabla\left(\nabla^2\Phi_\alpha\right)\Big\} = O(\mu^4). \tag{3.4}$$

From the dynamic boundary condition (2.8b) we obtain

$$\zeta = \Phi_\alpha + \mu^2 \left[(z_\alpha - \epsilon\zeta)\, \nabla \cdot (h\nabla\Phi_\alpha) + \tfrac{1}{2}\left(z_\alpha^2 - \epsilon^2\zeta^2\right) \nabla^2\Phi_\alpha \right] + O(\mu^4). \tag{3.5}$$

Taking the horizontal gradient of the preceding equation yields

$$\nabla\zeta = \nabla\Phi_\alpha + \mu^2 \nabla \left[(z_\alpha - \epsilon\zeta)\, \nabla \cdot (h\nabla\Phi_\alpha) + \tfrac{1}{2}\left(z_\alpha^2 - \epsilon^2\zeta^2\right) \nabla^2\Phi_\alpha \right]. \tag{3.6}$$

Following Chen & Liu (1995) we defined the horizontal velocity vector at $z = z_\alpha$ as

$$\begin{aligned} \boldsymbol{u}_\alpha &= -(\nabla\Phi)\,|_{z=z_\alpha} \\ &= -\nabla\Phi_\alpha - \mu^2 \left[(\nabla z_\alpha)\, \nabla \cdot (h\nabla\Phi_\alpha) + (z_\alpha \nabla z_\alpha)\, \nabla^2\Phi_\alpha \right] + O(\mu^4). \end{aligned} \tag{3.7}$$

Equations (3.4) and (3.6) can be rewritten in terms of ζ and $\boldsymbol{u}_\alpha$ as

$$\begin{aligned} \frac{\partial\zeta}{\partial t} + \nabla \cdot [(\epsilon\zeta + h)\boldsymbol{u}_\alpha] + \mu^2 \nabla \cdot \Bigg\{ &\left[\frac{z_\alpha^2}{2}(\epsilon\zeta + h) - \tfrac{1}{6}\left(h^3 + \epsilon^3\zeta^3\right) \right] \nabla\left(\nabla \cdot \boldsymbol{u}_\alpha\right) \\ &+ \left[z_\alpha(\epsilon\zeta + h) + \tfrac{1}{2}\left(h^2 - \epsilon^2\zeta^2\right) \right] \nabla\left[\nabla \cdot (h\boldsymbol{u}_\alpha)\right] \Bigg\} = O(\mu^4), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \nabla\zeta = -\boldsymbol{u}_\alpha - \mu^2 \Big\{ &(z_\alpha - \epsilon\zeta)\, \nabla\left[\nabla \cdot (h\boldsymbol{u}_\alpha)\right] + \tfrac{1}{2}\left(z_\alpha^2 - \epsilon^2\zeta^2\right) \nabla\left(\nabla \cdot \boldsymbol{u}_\alpha\right) \\ &- \nabla(\epsilon\zeta)\nabla \cdot (h\boldsymbol{u}_\alpha) - \tfrac{1}{2}\nabla\left(\epsilon^2\zeta^2\right) \nabla \cdot \boldsymbol{u}_\alpha \Big\} + O(\mu^4). \end{aligned} \tag{3.9}$$

For a given $z_\alpha(x, y)$ the above equations are the governing equations for ζ and $\boldsymbol{u}_\alpha$. Equation (3.8) is the continuity equation, while (3.9) is the momentum equation. We reiterate here that in both equations the parameter ϵ has been assumed to be of $O(1)$. Moreover, the water depth can also vary rapidly.

From (3.9) $\boldsymbol{u}_\alpha = -\nabla\zeta + O(\mu^2)$ and therefore, without reducing the order of magnitude of accuracy the velocity $\boldsymbol{u}_\alpha$ can be written in terms of ζ as

$$\begin{aligned} \boldsymbol{u}_\alpha = -\nabla\zeta + \mu^2 \Big\{ &(z_\alpha - \epsilon\zeta)\nabla\left[\nabla \cdot (h\nabla\zeta)\right] + \tfrac{1}{2}\left(z_\alpha^2 - \epsilon^2\zeta^2\right) \nabla(\nabla^2\zeta) \\ &- \nabla(\epsilon\zeta)\nabla \cdot (h\nabla\zeta) - \tfrac{1}{2}\nabla\left(\epsilon^2\zeta^2\right) \nabla^2\zeta \Big\} + O(\mu^4). \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8) yields the governing equation for the free-surface displacement ζ:

$$\begin{aligned} \frac{\partial\zeta}{\partial t} - \nabla \cdot [(\epsilon\zeta + h)\nabla\zeta] + \mu^2 \nabla \cdot \Big\{ &\tfrac{1}{6}\left(h^3 + \epsilon^3\zeta^3\right) \nabla\left(\nabla^2\zeta\right) - \tfrac{1}{2}\left(h^2 - \epsilon^2\zeta^2\right) \\ &\nabla\left[\nabla \cdot (h\nabla\zeta)\right] - (\epsilon\zeta + h)\nabla\left[\epsilon\zeta\nabla \cdot (h\nabla\zeta) + \tfrac{1}{2}\epsilon^2\zeta^2\nabla^2\zeta\right] \Big\} = O(\mu^4). \end{aligned} \tag{3.11}$$

It is interesting to note that z_α disappears in the equation for ζ. This is different from the equivalent depth-averaged equations for water waves, which depend on z_α (e.g. Chen & Liu 1995).

In the case that the bottom is flat, $h =$ constant, the governing equation can be simplified to

$$\begin{aligned} \frac{\partial\zeta}{\partial t} - \nabla \cdot [(\epsilon\zeta + h)\nabla\zeta] + \mu^2 \nabla \cdot \Big\{ &\tfrac{1}{6}\left(\epsilon^3\zeta^3 + 3\epsilon^2\zeta^2 h - 2h^3\right) \nabla\left(\nabla^2\zeta\right) \\ &- (\epsilon\zeta + h)\nabla\left[\epsilon\zeta\left(h + \tfrac{1}{2}\epsilon\zeta\right)\nabla^2\zeta\right] \Big\} = O(\mu^4). \end{aligned} \tag{3.12}$$

For the one-dimensional case, i.e. $\partial/\partial y = 0$, one obtains

$$\begin{aligned} \frac{\partial\zeta}{\partial t} - \frac{\partial}{\partial x}\left[(\epsilon\zeta + h)\frac{\partial\zeta}{\partial x}\right] + \mu^2 \frac{\partial}{\partial x} \Bigg\{ &\tfrac{1}{6}\left(\epsilon^3\zeta^3 + 3\epsilon^2\zeta^2 h - 2h^3\right) \frac{\partial^3\zeta}{\partial x^3} \\ &- (\epsilon\zeta + h)\frac{\partial}{\partial x}\left[\epsilon\zeta\left(h + \tfrac{1}{2}\epsilon\zeta\right)\frac{\partial^2\zeta}{\partial x^2}\right] \Bigg\} = O(\mu^4). \end{aligned} \tag{3.13}$$

Parlange *et al.* (1984) derived a set of perturbation equations in terms of the total water depth. The relation between Parlange *et al.*'s equations and (3.13) can be explained as follows. By letting H be the total water depth,

$$H = \epsilon\zeta + h, \tag{3.14}$$

one can rewrite (3.13) as

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) + \mu^2\frac{\partial}{\partial x}\left\{\frac{H}{6}\left[\epsilon^2\zeta^2 + 2h(\epsilon\zeta - h)\right]\frac{\partial^3 H}{\partial x^3} - \frac{H}{2}\frac{\partial}{\partial x}\left[(H^2 - h^2)\frac{\partial^2 H}{\partial x^2}\right]\right\} = O(\mu^4), \tag{3.15}$$

which can be further simplified, after some mathematical manipulations, to

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) - \mu^2\frac{1}{3}\frac{\partial^2}{\partial x^2}\left(H^3\frac{\partial^2 H}{\partial x^2}\right) = O(\mu^4). \tag{3.16}$$

Expanding H in terms of the small parameter μ^2 as

$$H = H_0 + \mu^2 H_1 + O(\mu^4), \tag{3.17}$$

we obtain the leading-order equations for H:

$O(1)$:

$$\frac{\partial H_0}{\partial t} - \frac{\partial}{\partial x}\left(H_0\frac{\partial H_0}{\partial x}\right) = 0, \tag{3.18}$$

$O(\mu^2)$:

$$\frac{\partial H_1}{\partial t} - \frac{\partial^2}{\partial x^2}(H_0 H_1) = \mu^2\frac{1}{3}\frac{\partial^2}{\partial x^2}\left(H_0^3\frac{\partial^2 H_0}{\partial x^2}\right). \tag{3.19}$$

Equations (3.18) and (3.19) agree with the one-dimensional equations derived by Parlange *et al.* (1984). As noted in § 1, Dagan's (1967) equations also agree with (3.18) and (3.19) after some typographic errors are corrected. Furthermore, the leading-order equation, (3.18), is the dimensionless form of (1.1) under the Dupuit approximation.

Since the free-surface displacement is assumed to be finite, the governing equations for ζ, (3.11), (3.12) or (3.13), are highly nonlinear. The highest order of differentiation is fourth order. To understand these equations and the effects of the shallowness of the aquifer, in the following sections the free-surface displacement will be assumed to be small to different degrees.

4. Boussinesq approximation

As mentioned in § 3 for some physical situations the free-surface displacement is small in comparison with depth, so we can assume that

$$O(\epsilon) = O(\mu^2) \ll 1, \tag{4.1}$$

and the governing equation for ζ, (3.11), can be simplified to

$$\frac{\partial \zeta}{\partial t} - \nabla\cdot\left[(\epsilon\zeta + h)\nabla\zeta\right] - \mu^2\nabla\cdot\left\{\tfrac{1}{2}h^2\nabla\left[\nabla\cdot(h\nabla\zeta)\right] - \tfrac{1}{6}h^3\nabla\left(\nabla^2\zeta\right)\right\} = O(\mu^4, \mu^2\epsilon). \tag{4.2}$$

This equation is equivalent to the Boussinesq equations for water waves. For the simplest one-dimensional case in which the water depth is a constant, (4.2) can be simplified to

$$\frac{\partial \zeta}{\partial t} - \epsilon \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h^3 \frac{\partial^4 \zeta}{\partial x^4} = 0. \tag{4.3}$$

The leading-order terms of the above equation are

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{1}{h}\frac{\partial \zeta}{\partial t} + O(\epsilon, \mu^2). \tag{4.4}$$

Hence, the last term in (4.3) can be represented by

$$\frac{\partial^4 \zeta}{\partial x^4} = \frac{1}{h}\frac{\partial^3 \zeta}{\partial x^2 \partial t} + O(\epsilon, \mu^2) = \frac{1}{h^2}\frac{\partial^2 \zeta}{\partial t^2} + O(\epsilon, \mu^2). \tag{4.5}$$

The alternative forms of (4.3) are

$$\frac{\partial \zeta}{\partial t} - \epsilon \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h \frac{\partial^2 \zeta}{\partial t^2} = 0, \tag{4.6a}$$

$$\frac{\partial \zeta}{\partial t} - \epsilon \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h^2 \frac{\partial^3 \zeta}{\partial x^2 \partial t} = 0. \tag{4.6b}$$

The corresponding dimensional form of these equations can be derived by substituting the dimensional variables, (2.7), into (4.3), (4.6*a*) and (4.6*b*). Thus

$$\frac{n_e}{K}\frac{\partial \zeta}{\partial t} - \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h_0\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}h_0^3 \frac{\partial^4 \zeta}{\partial x^4} = 0, \tag{4.7a}$$

$$\frac{n_e}{K}\frac{\partial \zeta}{\partial t} - \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h_0\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}h_0 \left(\frac{n_e}{K}\right)^2 \frac{\partial^2 \zeta}{\partial t^2} = 0, \tag{4.7b}$$

$$\frac{n_e}{K}\frac{\partial \zeta}{\partial t} - \frac{\partial}{\partial x}\left(\zeta \frac{\partial \zeta}{\partial x}\right) - h_0\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\frac{h_0^2 n_e}{K} \frac{\partial^3 \zeta}{\partial x^2 \partial t} = 0. \tag{4.7c}$$

Some of the fundamental characteristics of (4.7*a–c*) are discussed in §4.1.

4.1. *Linear diffusive waves*

For very small-amplitude motions, $O(\epsilon) \ll O(\mu^2) \ll 1$, the linearized versions of (4.3), (4.6*a*) and (4.6*b*) can be written as

$$\frac{\partial \zeta}{\partial t} - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h^3 \frac{\partial^4 \zeta}{\partial x^4} = 0, \tag{4.8a}$$

$$\frac{\partial \zeta}{\partial t} - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h \frac{\partial^2 \zeta}{\partial t^2} = 0, \tag{4.8b}$$

$$\frac{\partial \zeta}{\partial t} - h\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\mu^2 h^2 \frac{\partial^3 \zeta}{\partial x^2 \partial t} = 0. \tag{4.8c}$$

We remark here that similar equations can be obtained by considering different physical effects. For instance, Parlange & Brutsaert (1987) derived an equation for free-surface groundwater flow with a capillary correction, which has the same form as (4.8*c*). If the flow is periodic in time with a frequency ω, the leading-order terms of (4.8*a–c*) represent a damped wave motion. The third term, which is of $O(\mu^2)$, modifies the phase speed and spatial damping rate by including the depth effect. Assuming that the periodic solution can be written as

$$\zeta = a_0\, e^{i(kx-\omega t)}, \tag{4.9}$$

and substituting into (4.8) yields

$$-\,i\omega + hk^2 - \frac{1}{3}\mu^2 \begin{bmatrix} h^3k^4 \\ -\omega^2 h \\ i\omega h^2 k^2 \end{bmatrix} = 0, \tag{4.10a–c}$$

which can be interpreted as the dispersion relation for the system. The parameter k is the root of the dispersion relation and can be a complex quantity; the real part denotes the wavenumber and the imaginary part the spatial damping rate in the direction of wave propagation. Specifically (4.10a–c) can be solved for k^2 as

$$k^2 = \begin{cases} \dfrac{3}{2h^2\mu^2}\left[1-\left(1-\tfrac{4}{3}i\omega\mu^2 h\right)^{1/2}\right] & (4.11a) \\ \dfrac{1}{h}\left(i\omega - \tfrac{1}{3}\mu^2 h\omega^2\right) & (4.11b) \\ \dfrac{i\omega}{h\left(1-\tfrac{1}{3}i\omega\mu^2 h\right)}, & (4.11c) \end{cases}$$

where the nonsensical solution for k^2 in (4.11a) has been omitted.

The leading-order solution, as $\mu^2 \to 0$, of (4.11a–c) can be expressed as

$$k = (1+i)\left(\frac{\omega}{2h}\right)^{1/2}, \tag{4.11}$$

and the corresponding dimensional form is

$$k = \left(\frac{n_e}{K\, h_0}\right)^{1/2}(1+i)\left(\frac{\omega}{2}\right)^{1/2}. \tag{4.12}$$

For a surface wave with a given frequency ω, (4.11) or (4.12) implies that the wave will be damped out within a few wavelengths, because the imaginary part of (4.11) or (4.12), which is the spatial damping coefficient, is of the same order of magnitude as the real part. The dispersion relations in (4.11a–c) can also be rewritten as

$$\omega = \begin{cases} -ihk^2\left(1-\tfrac{1}{3}\mu^2h^2k^2\right) & (4.14a) \\ \dfrac{3i}{2\mu^2 h}\left[1 \pm \left(1+\tfrac{4}{3}\mu^2h^2k^2\right)^{1/2}\right] & (4.14b) \\ \dfrac{-ihk^2}{1+\tfrac{1}{3}\mu^2h^2k^2}. & (4.14c) \end{cases}$$

When the wavenumber is given, the real part of (4.14a–c) represents the wave frequency. The imaginary part denotes the temporal damping rate, if negative. If the imaginary part of ω, Im(ω), is positive, the corresponding equation is unstable. It is quite obvious that from (4.14a) Im(ω) $\leqslant 0$ only if $\mu^2k^2 \leqslant 3$. Hence, (4.8a) is conditionally stable. From (4.14b) one of the imaginary roots of ω is always positive. Thus, (4.8b) is unstable. On the other hand, the imaginary root of (4.14c) is always negative. Consequently, (4.8c) is unconditionally stable. In the remainder of this paper, (4.6b), which is the nonlinear version of (4.8c), will be used in the discussions.

5. Perturbation solutions for the tide-induced groundwater flow

In this section, we shall try to obtain a perturbation solution for (4.6*b*) subject to the forcing of a periodic tide. To simulate the effect of tide on the groundwater table, we assume that at the position $x = 0$ the aquifer is in contact with an ocean in which the water level undergoes periodic oscillations. So, we impose the following boundary condition:

$$\zeta(0, t) = e^{i\omega t}. \tag{5.1}$$

The aquifer extends to infinity in the x-direction, and the surface elevation becomes horizontal as $x \to \infty$, i.e.

$$\frac{\partial \zeta(x, t)}{\partial x} = 0, \qquad x \to \infty. \tag{5.2}$$

The initial free-surface displacement is set to be zero, i.e.

$$\zeta(x, 0) = 0. \tag{5.3}$$

By the Boussinesq approximation, (4.1), we can express μ^2 in terms of ϵ as

$$\mu^2 = \alpha\epsilon, \tag{5.4}$$

where α is a quantity of $O(1)$. Now we seek the solution of (4.6*b*) subject to the boundary conditions (5.1) and (5.2) in the following perturbation form:

$$\zeta = \zeta_0 + \epsilon\zeta_1 + O(\epsilon^2). \tag{5.5}$$

Substituting (5.5) into (4.6*b*), (5.1) and (5.2) and collecting terms of like powers of ϵ, one can obtain the following set of equations for ζ of different of orders of magnitude:

$O(\epsilon^0)$:

$$\frac{\partial \zeta_0}{\partial t} - h\frac{\partial^2 \zeta_0}{\partial x^2} = 0, \qquad 0 < x < \infty, \tag{5.6a}$$

$$\zeta_0(0, t) = e^{i\omega t}, \tag{5.6b}$$

$$\frac{\partial \zeta_0(x, t)}{\partial x} = 0, \qquad x \to \infty. \tag{5.6c}$$

$O(\epsilon)$:

$$\frac{\partial \zeta_1}{\partial t} - h\frac{\partial^2 \zeta_1}{\partial x^2} = \frac{\partial}{\partial x}\left(\zeta_0 \frac{\partial \zeta_0}{\partial x}\right) + \frac{1}{3}\alpha h^2 \frac{\partial^3 \zeta_0}{\partial x^2 \partial t}, \qquad 0 < x < \infty, \tag{5.7a}$$

$$\zeta_1(0, t) = 0, \tag{5.7b}$$

$$\frac{\partial \zeta_1(x, t)}{\partial x} = 0, \qquad x \to \infty. \tag{5.7c}$$

Because the only driving force of the flow comes from the periodic tide at the boundary, we can expect that the motions in the aquifer will approach a periodic state eventually. Therefore, we shall pursue a periodic solution for ζ. As shown in § 4.1, the leading-order periodic solution for ζ_0 is

$$\zeta_0 = e^{i(k_0 x - \omega t)}, \qquad k_0 = (1 + i)\left(\frac{\omega}{2h}\right)^{1/2}, \tag{5.8}$$

which represent a free-surface displacement decaying with distance from the boundary while oscillating in both space and time. The decay rate is of the same order magnitude as the wavenumber, because k_0 is a complex number and the imaginary part is equal to the real part.

Substituting (5.8) into the right-hand-side of (5.7a), one has

$$\frac{\partial \zeta_1}{\partial t} - h\frac{\partial^2 \zeta_1}{\partial x^2} = -k_0^2 e^{2i(k_0 x - \omega t)} - \tfrac{1}{4}\left(k_0 - k_0^*\right)^2 e^{i(k_0 - k_0^*)x} + \tfrac{1}{3} i\alpha h^2 k_0^2 \omega e^{i(k_0 x - \omega t)}, \quad 0 < x < \infty, \tag{5.9}$$

where the asterisk denotes the complex conjugate. There are three forcing terms for ζ_1: the first-harmonic, the second harmonic and the steady terms. While the first-harmonic term is due to the linear high-order derivative term, the steady and second-harmonic terms are from the nonlinear interactions of the leading-order waves. The first harmonic term is a secular term and resonates with the operator on the left-hand side. The particular solution for the secular term is $\zeta_1 \propto x e^{i(k_0 x - \omega t)}$. Therefore, the ratio between the $O(\epsilon)$ solution and the $O(\epsilon^0)$ solution, ζ_1/ζ_0, becomes unbounded in x. In order to eliminate the secular term, we expand the wavenumber in a series form

$$k = k_0 + \epsilon k_1 + O(\epsilon^2), \tag{5.10}$$

and take

$$\zeta_0 = e^{i(kx - \omega t)}. \tag{5.11}$$

Then, the leading-order equation, (5.6a), remains the same, but the governing equation in $O(\epsilon)$, (5.7a), becomes

$$\begin{aligned} \frac{\partial \zeta_1}{\partial t} - h\frac{\partial^2 \zeta_1}{\partial x^2} &= \frac{\partial}{\partial x}\left(\zeta_0 \frac{\partial \zeta_0}{\partial x}\right) + \frac{1}{3}\alpha h^2 \frac{\partial^3 \zeta_0}{\partial x^2 \partial t} - 2k_0 k_1 h \zeta_0, \\ &= -k_0^2 e^{2i(k_0 x - \omega t)} - \tfrac{1}{4}\left(k_0 - k_0^*\right)^2 e^{i(k_0 - k_0^*)x} \\ &\quad + \left(\tfrac{1}{3} i\alpha h^2 k_0^2 \omega - 2k_0 k_1 h\right) e^{i(k_0 x - \omega t)}, \qquad 0 < x < \infty. \end{aligned} \tag{5.12}$$

To eliminate the secular term, we must require

$$k_1 = \tfrac{1}{6} i\alpha k_0 h \omega, \tag{5.13}$$

which is also a complex number and represents the effects of finite depth on the wavenumber and damping rate. We remark here that (5.13) can also be obtained directly from (4.11a–c). Therefore, the equation for ζ_1 is now

$$\frac{\partial \zeta_1}{\partial t} - h\frac{\partial^2 \zeta_1}{\partial x^2} = -k_0^2 e^{2i(k_0 x - \omega t)} - \tfrac{1}{4}\left(k_0 - k_0^*\right)^2 e^{i(k_0 - k_0^*)x}, \qquad 0 < x < \infty. \tag{5.14}$$

Solving (5.14) with boundary conditions (5.7b) and (5.7c), we obtain

$$\zeta_1 = \frac{1}{2h}\left[e^{i(\sqrt{2}k_0 x - 2\omega t)} - e^{2i(k_0 x - \omega t)}\right] + \frac{1}{4h}\left[1 - e^{i(k_0 - k_0^*)x}\right]. \tag{5.15}$$

Because k_0 is complex number with positive imaginary part, the oscillating terms in (5.15) vanish far away from the interface of the ocean and aquifer, i.e.

$$\zeta_1 = \frac{1}{4h}, \qquad x \to \infty. \tag{5.16}$$

This implies that the groundwater level will rise in the entire aquifer due to the effect of the periodic tidal forcing. This phenomenon has been observed in the field (e.g. Nielsen 1990). We remark here that the rise of the groundwater level is similar to the mean free-surface setdown and setup in the shoaling and breaking of water waves.

In summary, the perturbation solution of the nonlinear wave equation (4.6*b*) subject to periodic tidal forcing is

$$\zeta = \mathrm{e}^{\mathrm{i}(kx-\omega t)} + \frac{\epsilon}{4h}\left\{2\left[\mathrm{e}^{\mathrm{i}(\sqrt{2}k_0x-2\omega t)} - \mathrm{e}^{2\mathrm{i}(k_0x-\omega t)}\right] + \left[1-\mathrm{e}^{\mathrm{i}(k_0-k_0^*)x}\right]\right\}, \tag{5.17a}$$

$$k = k_0\left(1+\tfrac{1}{6}\mathrm{i}\mu^2\omega h\right), \qquad k_0 = (1+\mathrm{i})\left(\frac{\omega}{2h}\right)^{1/2}. \tag{5.17b}$$

Similar perturbation solutions can also be obtained for (4.3) and (4.6*a*). The profiles of ζ along x at $\omega t = 0, \pi/2, \pi, 3\pi/2$ are presented in figure 1 with different parameters, according to the solution (5.17*a*) with $h = 1$. Figure 1 clearly demonstrates the nonlinear effect, which induces a rise of the mean water level in the aquifer. But the dominating mechanism inside the aquifer is diffusion. The μ^2-term affects the spatial variation of the free surface significantly.

The corresponding dimensional form of (5.17*a*) is

$$\zeta = a\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \frac{a^2}{4h_0}\left\{2\left[\mathrm{e}^{\mathrm{i}(\sqrt{2}k_0x-2\omega t)} - \mathrm{e}^{2\mathrm{i}(k_0x-\omega t)}\right] + \left[1-\mathrm{e}^{\mathrm{i}(k_0-k_0^*)x}\right]\right\}, \tag{5.18a}$$

where the dimensional wavenumber is defined as

$$k = k_0\left(1+\frac{\mathrm{i}}{6}\frac{n_e}{K}\omega h_0\right), \qquad k_0 = (1+\mathrm{i})\left(\frac{n_e\omega}{2Kh_0}\right)^{1/2}. \tag{5.18b}$$

Parlange *et al.* (1984) performed a set of laboratory experiments measuring the surface water elevations in porous media driven by an oscillating piston in a reservoir in contact with the porous media. The water level in the reservoir is oscillating with period equal to 35 s. The static water level in the flume is $h = 0.276$ m and the wave amplitude is $a = 0.09$ m. The ratio of the porosity to permeability is $n_e/K \approx 34.5\ \mathrm{s\,m^{-1}}$. They observed that steady periodic motions could be reached and presented their measurements of the steady oscillating water levels at four locations in the porous media. These experimental data are compared with our analytical solutions (5.18*a*) in figure 2. The agreement between the analytical solutions and the experimental data are very good. To demonstrate the effect of the high-order terms, the solutions with $\mu^2 = 0$, which have also been obtained by Parlange *et al.* (1984) and Nielsen (1990), are presented in the figure. The present solutions with the μ^2-term show much better agreement than that without it, especially far away from the boundary, $x = 0$.

6. Numerical solutions for the nonlinear equation

Because analytical solutions to the nonlinear diffusive wave equations are difficult to obtain except under some special situations like the periodic problem described in § 5, we have to employ numerical methods to solve the problem in general cases. To solve the nonlinear diffusive wave equation (4.6*b*), we employ a numerical scheme with a central finite difference method in space and a fourth-order Runge–Kutta integration technique in time. Numerical solutions are discussed in this section.

6.1. *Numerical scheme*

Equation (4.6*b*) can be rewritten as

$$\frac{\partial\psi}{\partial t} = F(\zeta) \tag{6.1a}$$

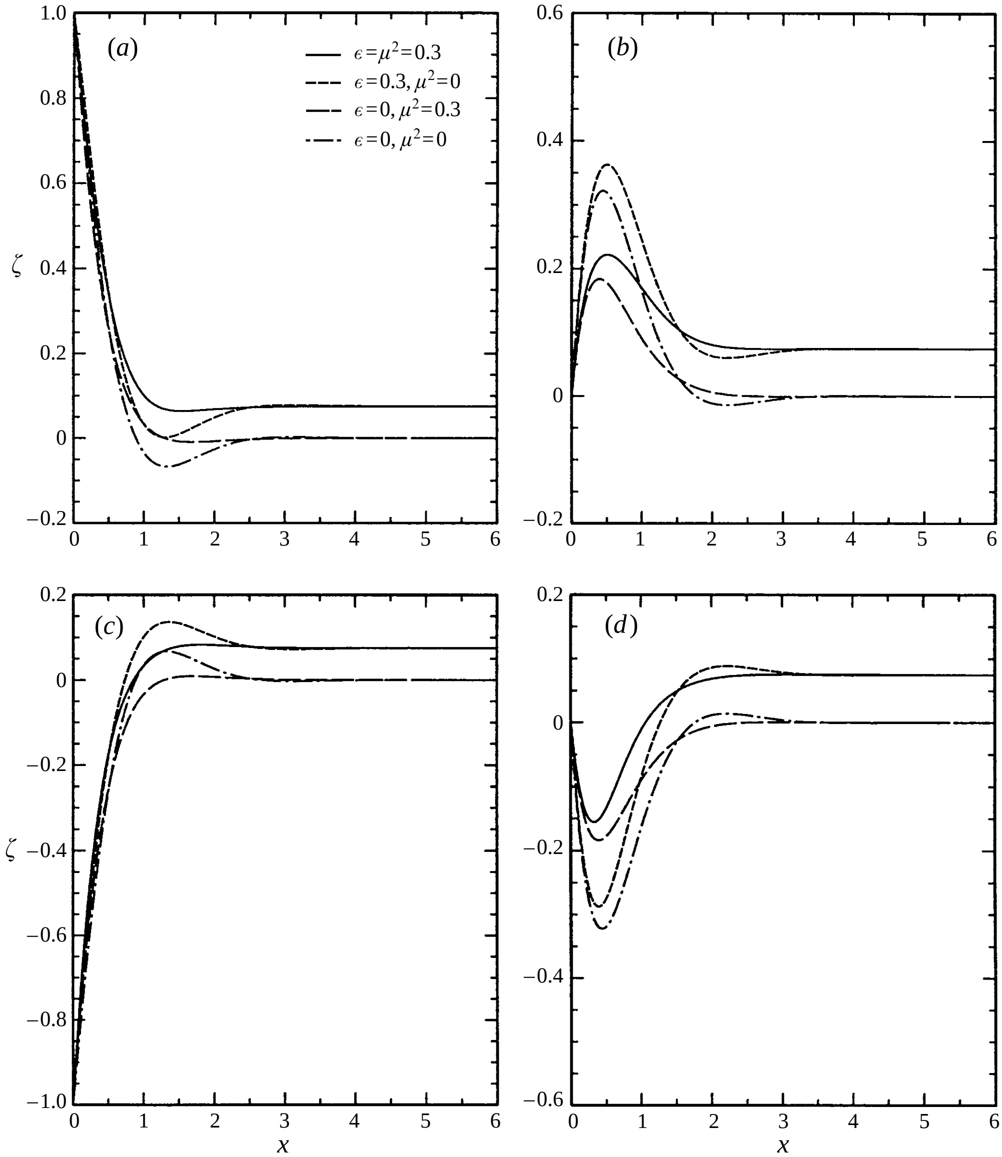

FIGURE 1. Analytical solutions of the nonlinear equations with different parameters: (*a*) $\omega t = 0$; (*b*) $\omega t = \pi/2$; (*c*) $\omega t = \pi$; (*d*) $\omega t = 3\pi/2$.

where

$$\psi = \left(1 - \frac{1}{3}\mu^2 h^2 \frac{\partial^2}{\partial x^2}\right)\zeta, \qquad F(\zeta) = \epsilon\left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \zeta\frac{\partial^2 \zeta}{\partial x^2}\right] - h\frac{\partial^2 \zeta}{\partial x^2}. \tag{6.1b}$$

Then, ζ can be regarded as a function of ψ, i.e.

$$\zeta(x,t) = f\,[\psi(x,t)]\,. \tag{6.2}$$

In order to integrate (6.1*a*) numerically, we employ a simple second-order central finite difference method to evaluate the spatial derivatives in (6.1*b*). By dividing the

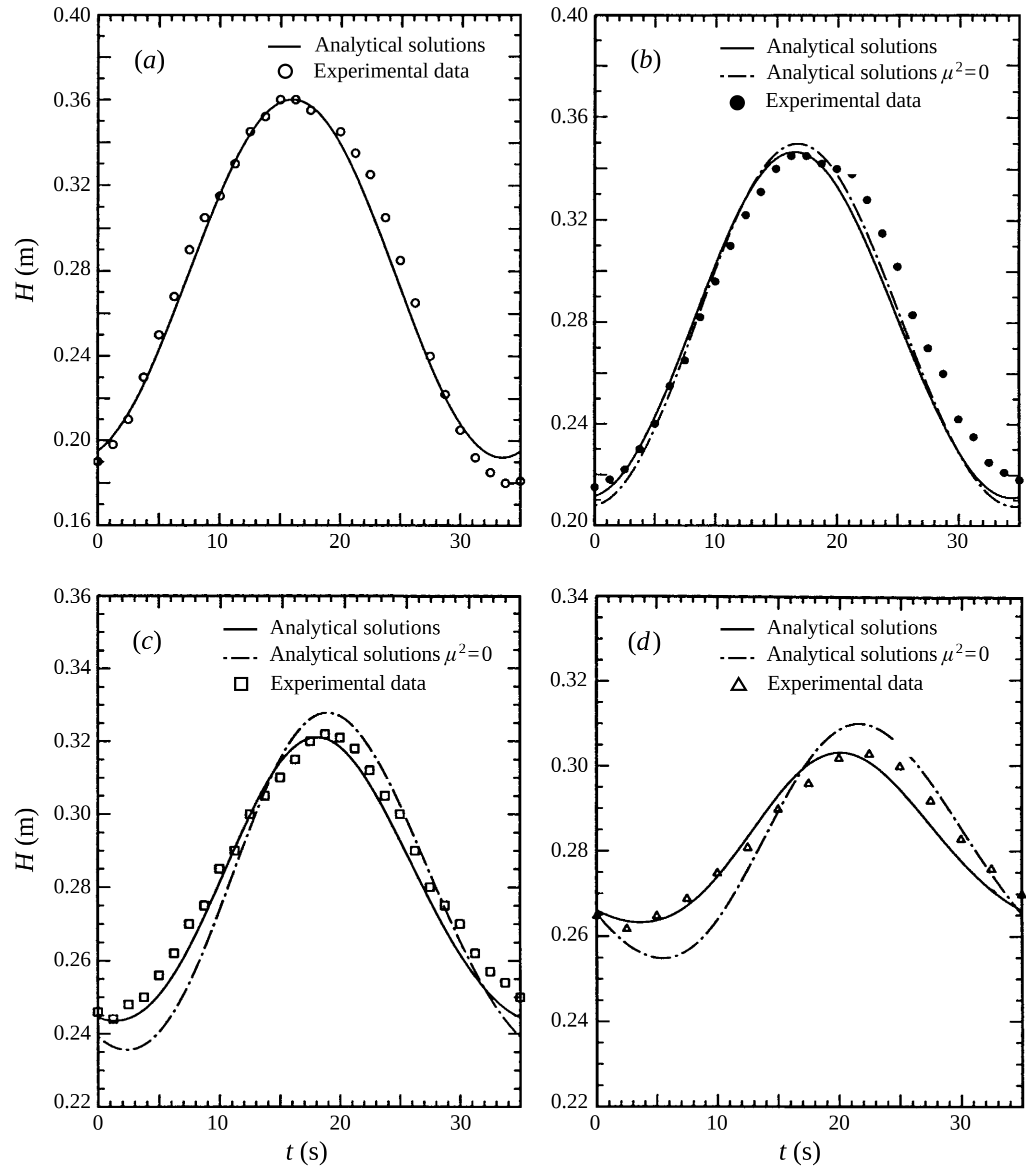

FIGURE 2. Water table in porous media at (*a*) $x = 0$ m; (*b*) $x = 0.05$ m; (*c*) $x = 0.18$ m; (*d*) $x = 0.335$ m.

x-domain, $0 \leqslant x \leqslant l$, into $N-1$ sections of equal length Δx, the finite difference formula for $\psi(x,t)$ can be written from (6.1*b*) as

$$\psi(x_n,t) = -\frac{\mu^2 h^2}{3(\Delta x)^2}\zeta(x_{n-1},t) + \left[1 + \frac{2\mu^2 h^2}{3(\Delta x)^2}\right]\zeta(x_n,t) - \frac{\mu^2 h^2}{3(\Delta x)^2}\zeta(x_{n+1},t), \qquad (6.3a)$$

where

$$x_n = n\,\Delta x, \qquad n = 0, 1, 2, \cdots, N. \qquad (6.3b)$$

Equation (6.3) can be readily used to obtain $\psi(x,\ t)$ when $\zeta(x,t)$ is known inside the domain and appropriate boundary conditions for ζ are provided. On the other hand, if $\psi(x,\, t)$ inside the domain and boundary conditions for ζ are given, (6.3) represents a set of linear equations for $\zeta(x_n, t)$. Hence, $\zeta(x,\, t)$ inside the domain can be found by solving the linear equations given by (6.3). For the sake of convenience, we express this process of obtaining ζ from ψ in the form of (6.2).

For the time integration of (6.1*a*) we use the fourth-order Runge–Kutta method (RK4) to evaluate ψ at the next time level $t + \Delta t$, as follows (Press *et al.* 1992):

$$\psi(x_n, t + \Delta t) = \psi(x_n, t) + \tfrac{1}{6}\left[r_1(x_n, t) + 2r_2(x_n, t) + 2r_3(x_n, t) + r_4(x_n, t)\right]\Delta t, \qquad (6.4a)$$

in which

$$r_1(x_n, t) = F\left[\zeta(x_n, t)\right], \qquad (6.4b)$$

$$r_2(x_n, t) = F\left[\zeta_1(x_n, t)\right], \qquad \zeta_1(x_n, t) = f\left[\psi(x_n, t) + \tfrac{1}{2} r_1(x_n, t)\Delta t\right], \qquad (6.4c)$$

$$r_3(x_n, t) = F\left[\zeta_2(x_n, t)\right], \qquad \zeta_2(x_n, t) = f\left[\psi(x_n, t) + \tfrac{1}{2} r_2(x_n, t)\Delta t\right], \qquad (6.4d)$$

$$r_4(x_n, t) = F\left[\zeta_3(x_n, t)\right], \qquad \zeta_3(x_n, t) = f\left[\psi(x_n, t) + r_3(x_n, t)\Delta t\right]. \qquad (6.4e)$$

We reiterate here that ζ_1, ζ_2 and ζ_3 in (6.4) are obtained by solving a set of linear equation similar to (6.3). The process is represented by $f(\psi)$. The function $F(\zeta)$ in (6.4) denotes the evaluation of the F-term defined in (6.1*b*) by the central finite difference scheme.

After obtaining $\psi(x,\, t + \Delta t)$ from (6.4), we could find $\zeta(x,\, t + \Delta t)$ by solving the set of linear equation, (6.3) at time level $t + \Delta t$. Boundary conditions on ζ are enforced through (6.3) in the process of the time integration.

Note that the truncation errors of the central finite difference scheme in space and the fourth-order Runge–Kutta integration method in time are of $O[(\Delta x)^2]$ and $O[(\Delta t)^5]$, respectively. In order to avoid the interference of the numerical discretization errors with the dissipation and nonlinearity in the original equation (4.6*b*), we require in the present numerical scheme that

$$\left[(\Delta x)^2, \Delta t\right] \ll \epsilon, \qquad \left[(\Delta x)^2, \Delta t\right] \ll \mu^2. \qquad (6.5)$$

Hence, the numerical truncation errors will be much smaller than the effects of the nonlinear and high-derivative terms in the nonlinear dispersive and diffusive wave equation. Because of the dissipative nature of the aquifer the wave will be damped out in several wavelengths, as demonstrated in § 4.1. In such a short distance, the effects of the cumulative numerical truncation error on the numerical solutions is negligible.

6.2. *Comparisons of numerical results with analytical solutions and experimental data*

To verify our numerical scheme, we first solve the nonlinear diffusive equation (4.6*b*) for the tide-induced groundwater flow as described in § 5. By taking the dimensionless physical parameters $h = 1$, $\omega = 2\pi$ and the numerical parameters, $\Delta t = 0.01$, $\Delta x = 0.01$, $l = 6$, the periodic numerical solutions for the water table in the aquifer are compared in figure 3 with the perturbation solutions for two set of the dimensionless parameters, ϵ and μ^2, at fixed time. Figure 3 demonstrates that the difference between the analytical solutions and the numerical solutions is negligible for the case $\epsilon = \mu^2 = 0.1$. The numerical results show slightly more diffusion than the perturbation solution when $\epsilon = \mu^2 = 0.3$.

Figure 4 presents the numerical solutions, the analytical solutions and the experimental data for the periodic water table at $x = 0.18$ and 0.335 m as a function of time t. In this numerical computation, we solved the dimensional nonlinear equation

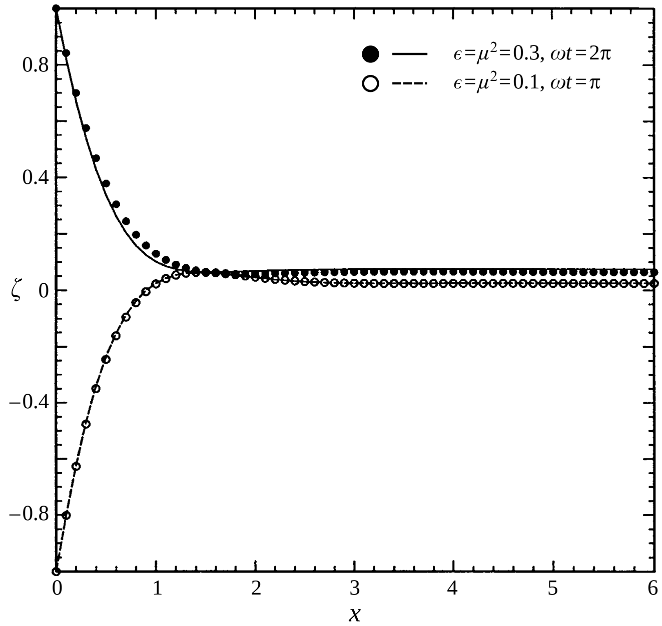

FIGURE 3. Comparison between numerial solutions and analtical solutions. The lines represent the pertubation solution and the symbols are numerical solutions.

(4.7c) and employed the following numerical parameters $\Delta t = 0.01$ s, $\Delta x = 0.005$ m, $l = 2$ m. Figure 4 demonstrates a reasonably good agreement among the numerical results, analytical solutions and the experimental data. We remark here that for Parlange *et al.*'s experimental conditions, $\mu^2 \approx 0.27$ and $\epsilon \approx 0.33$, the good agreement between experiment data and theoretical results for significant magnitudes of ϵ and μ^2 demonstrates the apparent robustness of theory.

6.3. *Reflection and transmission of a solitary wave by a porous breakwater*

In this subsection, the weakly nonlinear equations derived in §4 are employed to investigate the effectiveness of porous breakwaters. Consider a solitary wave incident normally on a porous breakwater with width b. The water depth is assumed to be constant, h_0. The objective is to find the relationship between the reflection and transmission coefficients and the properties of the porous materials used in the breakwater. In the free water regions, upstream and downstream of the breakwater, the Boussinesq equations are employed (e.g. Whitham 1973):

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left[(h_0 + \zeta)\, u\right] = 0, \tag{6.6a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} - \frac{1}{3}h_0^2\frac{\partial^3 u}{\partial x^2 \partial t} = 0, \tag{6.6b}$$

where u the depth-averaged velocity, and all the physical quantities are in the dimensional form. Inside the porous breakwater, the following dimensional equations are used:

$$\frac{n_e}{K}\frac{\partial \zeta}{\partial t} - \frac{\partial}{\partial x}\left(\zeta\frac{\partial \zeta}{\partial x}\right) - h_0\frac{\partial^2 \zeta}{\partial x^2} - \frac{1}{3}\frac{h_0^2 n_e}{K}\frac{\partial^3 \zeta}{\partial x^2 \partial t} = 0, \tag{6.7a}$$

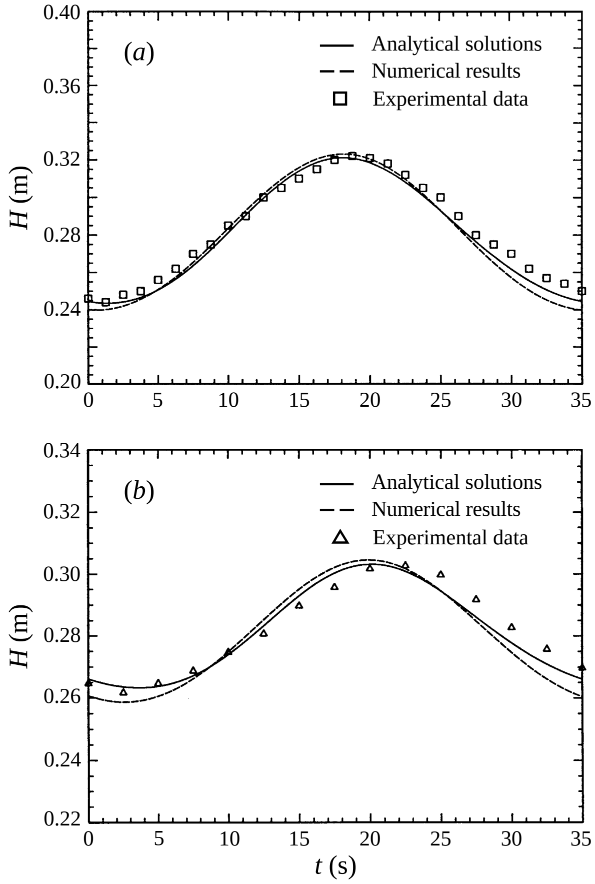

FIGURE 4. Comparison among numerical results, analytical solutions and experimental data (Parlange *et al.* 1984): (*a*) $x = 0.18\,\mathrm{m}$; (*b*) $x = 0.335\,\mathrm{m}$.

$$u + K\frac{\partial \zeta}{\partial x} - \frac{1}{3}h_0^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{6.7b}$$

where (6.7*a*) is the same as (4.8*c*) and the equation for the depth-averaged velocity u, (6.7*b*), has been obtained by averaging (3.10) over the entire water depth.

The permeability of the porous breakwater can be expressed as (Madsen 1974; Vidal *et al.* 1988)

$$K = \frac{g}{\left(\dfrac{c_1 \nu}{d^2 n_e} + \dfrac{c_2}{d n_e^2}U_c\right)}, \tag{6.8}$$

where U_c is a characteristic velocity in the porous breakwater (see the Appendix), ν the kinematic viscosity of pore water, d the diameter of the gravel comprising the breakwater, and c_1 and c_1 are empirical coefficients,

$$c_1 = \alpha\frac{(1-n_e)^3}{n_e}, \qquad c_2 = \beta\frac{(1-n_e)}{n_e}, \tag{6.9}$$

in which α and β are dimensionless constants. The two terms in the denominator of (6.8) represent the effects of the laminar and turbulent resistance, respectively, in the porous medium.

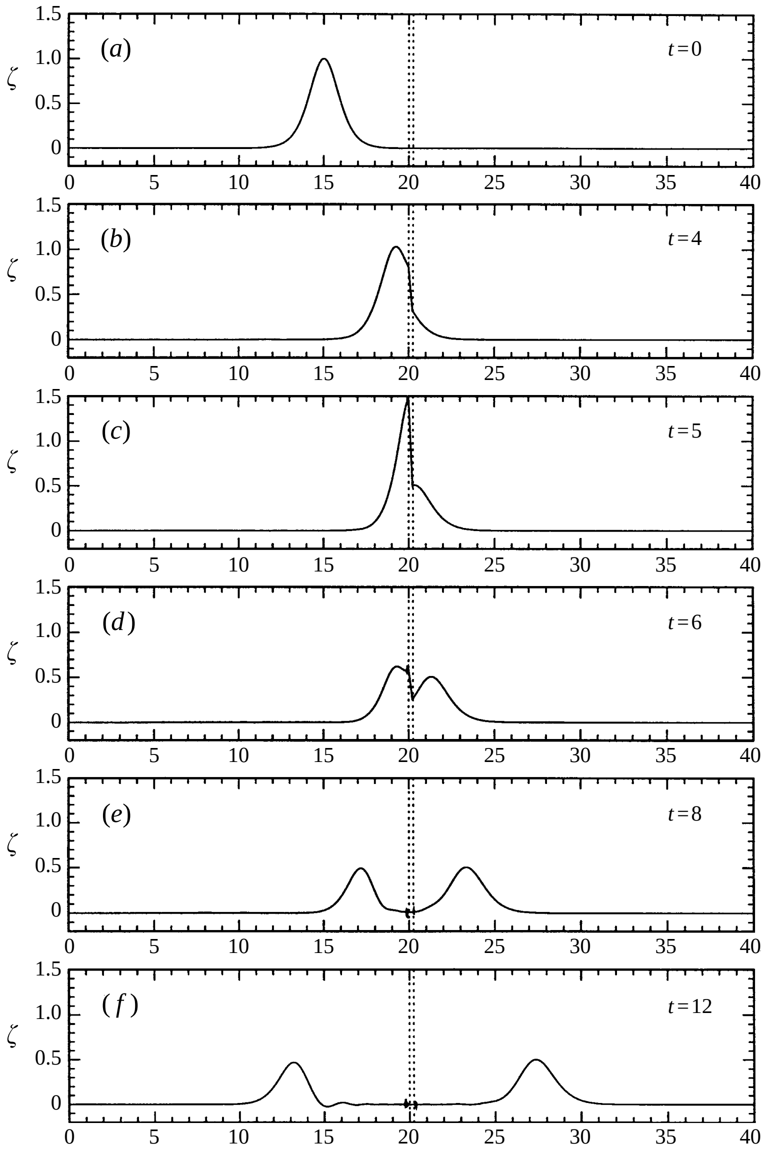

FIGURE 5. Evolution of a solitary wave through a porous breakwater.

To facilitate the solutions of wave motions in different regions, interfacial boundary conditions are enforced in the present numerical scheme, such that ζ and u as well as their spacial derivatives are continuous across the interfaces between the shallow water and the breakwater, respectively, i.e.

$$\zeta|_{+} = \zeta|_{-}, \qquad u|_{+} = u|_{-}, \tag{6.10a}$$

$$\left.\frac{\partial \zeta}{\partial x}\right|_{+} = \left.\frac{\partial \zeta}{\partial x}\right|_{-}, \qquad \left.\frac{\partial u}{\partial x}\right|_{+} = \left.\frac{\partial u}{\partial x}\right|_{-}, \tag{6.10b}$$

where $\pm$ denote the left-hand side and the right-hand side of the interface, respectively.

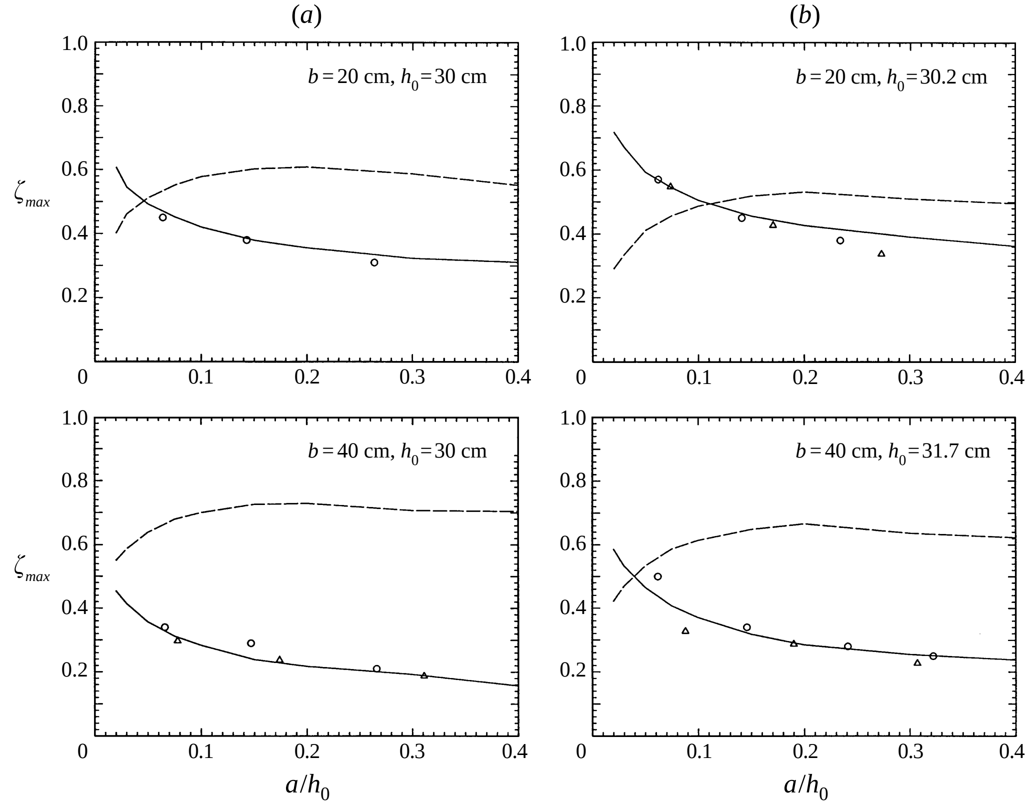

FIGURE 6 (a, b). For caption see facing page.

A similar numerical scheme to that in § 6.1 is employed to integrate the Boussinesq equations, (6.6), simultaneously with the nonlinear wave equations, (6.7a). The shallow water regions are assumed to extend to infinity so that the boundary effects can be neglected.

Figure 5 shows the evolution of a solitary wave through a breakwater consisting of gravel with $d = 2.34$ cm, $n_e = 0.44$, $\alpha = 1092$, $\beta = 0.81$, $b = 20$ cm, $a = 3$ cm, $h_0 = 30$ cm. The x-coordinate is scaled by $(h_0^3/a)^{1/2}$, ζ by a and t by $h_0/(ga)^{1/2}$. The initial solitary wave profile is taken as (Whitham 1973)

$$\zeta(x, 0) = a\,\mathrm{sech}^2\left\{\left(\frac{3}{4}\right)^{1/2}\left[\left(\frac{a}{h_0^3}\right)^{1/2} x - 15\right]\right\}, \tag{6.11a}$$

$$u = \left(\frac{g}{h_0}\right)^{1/2}\left[\left(1 + \frac{1}{2}\frac{a}{h_0}\right)\zeta(x, 0) + \frac{1}{h_0}\zeta^2(x, 0)\right]. \tag{6.11b}$$

The incident solitary wave is partially transmitted through and partially reflected from the porous breakwater. The shapes of both the transmitted and reflected waves are similar to solitary waves. Small trailing waves are developed in both reflected and transmitted waves because of frequency dispersion.

Vidal *et al.* (1988) performed a series of laboratory experiments on solitary wave transmission through porous breakwaters. The permeable breakwaters had rectangular

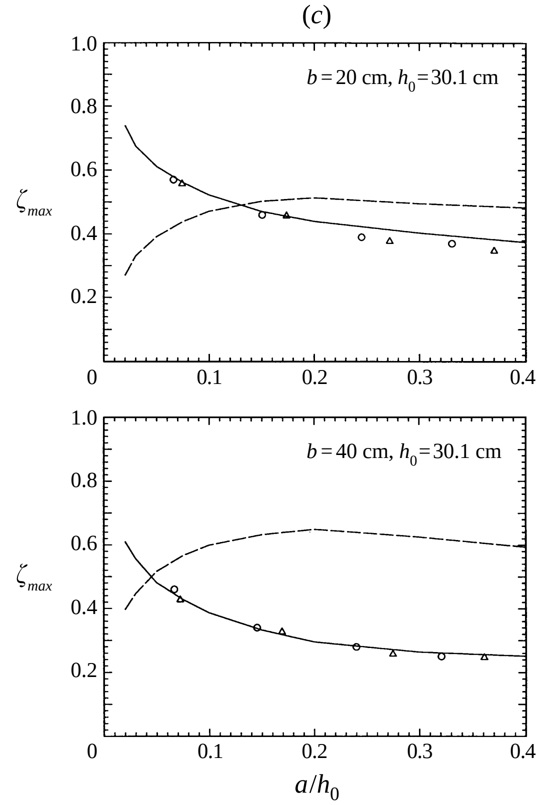

FIGURE 6. Transmission and reflection of solitary waves by permeable breakwaters: ——, computed maximum transmitted wave amplitude; – – –, computed maximum reflected wave amplitude, and the symbols represent the experimental data from Vidal *et al.* (1988) for maximum transmitted wave amplitude (*a*) gravel, $d = 1.43$ cm; (*b*) gravel, $d = 2.43$ cm; (*c*) permeable cubes with sidelength = 3.15 cm.

form and were 20 to 40 cm thick. The tests were carried with gravel of $d = 1.43, 2.43$ cm and small cubic blocks of 3.15 cm side length. The measured porosities were $n_e = 0.44$ for gravel and $n_e = 0.42$ for cubes. The water depth in the test flume varied from 25 to 30 cm. The maximum wave amplitude was measured 200 cm downstream of the porous structures. By taking $\alpha = 1092$ and $\beta = 0.81$ which were recommended by Vidal *et al.* (1988), the numerical results for the transmitted and reflected wave amplitudes by present model are compared with Vidal *et al.*'s experiment data in figures 6(*a*), 6(*b*) and 6(*c*), where the transmitted wave amplitude is scaled by the incident wave amplitude. The scaled transmitted wave amplitude decreases as the nonlinear effect and the thickness of the breakwater increase for all the cases. The present numerical results are in very good with the experiment data. The reflected wave amplitude increases with a/h_0 when $a/h_0 < 0.2$. As a/h_0 further increases, nonlinear effects becomes more significant and so does the frequency dispersion induced by the breakwater. The reflected wave consists of a dominant wave with some trailing waves (see figure 5). The amplitude of the trailing waves increases with a/h_0. Hence, the dominant reflected wave amplitude then decreases with the incident wave amplitude.

7. Concluding remarks

A two-dimensional fully nonlinear long-wave equation has been derived to describe flow in a phreatic aquifer. Using the Boussinesq approximation a simplified equation is derived for weakly nonlinear, dispersive and diffusive waves. The model equation has been employed to investigate one-dimensional tide-induced fluctuation in aquifers and the effectiveness of porous breakwaters. The numerical results of the present model equation demonstrate good agreement with laboratory experiments. Although only the one-dimensional problems are discussed in the paper, the model equation can be applied to two-dimensional problems.

Appendix. Characteristic velocity in a porous breakwater

The resistance force in a porous breakwater is more of the Dupuit–Forchheimer type than of Darcy type (e.g. Madsen 1974). Thus

$$F_R = \left(\frac{c_1 \nu}{d^2 n_e} + \frac{c_2}{d n_e^2} u \right) u. \tag{A 1}$$

It is necessary to linearize the resistance formula in order to adapt the present theory. As an approximation, we employ a characteristic velocity U_c and express the resistance force as

$$\hat{F}_R = \left(\frac{c_1 \nu}{d^2 n_e} + \frac{c_2}{d n_e^2} U_c \right) u. \tag{A 2}$$

To find U_c, we require that the linearized and nonlinear resistance forces do the same amount of work over the entire porous breakwater, i.e.

$$\int_0^b F_R u \, \mathrm{d}x = \int_0^b \hat{F}_R u \, \mathrm{d}x. \tag{A 3}$$

Since the horizontal length scale of the solitary wave is much longer than the width of the porous breakwater, it is reasonable to assume that the velocity inside the breakwater varies linearly (Madsen 1974). Hence

$$u = (1 + \gamma x) u_0, \tag{A 4}$$

where γ is a constant and u_0 is the velocity at the upstream edge of the breakwater, $x = 0$. Substituting (A 4) into (A 3) and integrating the resulting integrals yield

$$U_c = \frac{3}{4} \frac{(u_b^4 - u_0^4)}{(u_b^3 - u_0^3)}, \tag{A 5}$$

where u_b is the velocity at the downstream face of the breakwater, $x = b$. Therefore U_c is a constant in the entire breakwater, but it changes with time since u_b and u_0 are functions of time.

REFERENCES

Bear, J. 1972 *Dynamics of Fluid in Porous Media.* American Elsevier, New York.

Chen, Y. & Liu, P. L.-F. 1995 Modified Boussinesq equations and associated parabolic models for water wave propagation. *J. Fluid Mech.* **288**, 351–381.

Dagan, G. 1967 Second-order theory of shallow free surface flow in porous media. *Q. J. Mech. Appl. Maths* **20**, 517–526.

Knight, J. H. 1982 Steady periodic flow through a rectangular dam. *Water Resour. Res.* **12**, 1222–1224.

Losada, I. J., Losada, M. A. & Martin, F. L. 1995 Experimental study of wave-induced flow in a porous structure. *Coastal Engng* **26**, 77–98.

Madsen, O. S. 1974 Wave transmission through porous structures. *J. Waterway, Harbors, Coastal Engng Div. ASCE* **100** (WW3), 169–188.

Nielsen, P. 1990 Tidal dynamics of the water table in beaches. *Water Resour. Res.* **26**, 2127–2134.

Nielsen, P., Aseervatham, R., Fenton, J. D. & Perrochet, P. 1996 Groundwater waves in aquifer of intermediate depths. *Adv. Water Resour.* **20**, 37–43.

Parlange, J.-Y. & Brutsaert, W. 1987 A capillarity correction for free surface flow of groundwater. *Water Resour. Res.* **23**, 805–808.

Parlange, J.-Y., Stagnitti, F., Starr, J. L. & Braddock, R. D. 1984 Free-surface flow in porous media and periodic solution of the shallow-flow approximation. *J. Hydrology* **70**, 251–263.

Philip, J. R. 1973 Periodic nonlinear diffusion: an integral relation and its physical consequences. *Austral. J. Phys.* **26**, 513–519.

Press, W. H., Teukolsky, S. A., Vettering, W. T. & Flannery, B. P. 1992 *Numerical Recipes in Fortran, The Art of Scientific Computing,* 2nd edn. Cambridge University Press.

Vidal, C., Losada, M. A., Medina, R. & Rubio, J. 1988 Solitary wave transmission through porous breakwaters. *Proc. 21st Intl Coastal Engng Conf., Costa del Sol – Malaga, Spain*, pp. 1073–1083.

Whitham, G. B. 1973 *Linear and Nonlinear Waves*. John Wiley & Sons.